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ω-LINEAR VECTOR FIELDS ON MANIFOLDS

BY

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ABSTRACT. The classical study of a flow near a fixed point is generalized by composing, at each point in the manifold, the flow derivative with a parallel translation back along the flow. Circumstances under which these compositions form a one-parameter group are studied. From the point of view of the linear frame bundle, the condition is that the canonical lift commute with its horizontal part (with respect to some metric connection). The connection form applied to the lift coincides with the infinitesimal generator of the one-parameter group. Analysis of this matrix provides dynamical information about the flow. For example, if such flows are equicontinuous, they have uniformly bounded derivatives and therefore the enveloping semigroup is a Lie transformation group. Subclasses of ergodic, minimal, and weakly mixing flows with integral invariants are determined according to the eigenvalues of the matrices. Such examples as Lie algebra flows, infinitesimal affine transformations, and the geodesic flows on manifolds of constant negative curvature are examined.

0. Introduction. This work grew out of an attempt to generalize the classical consideration of a vector field on a manifold through analysis of the flow derivatives. Classically, if $\{X_t \mid t \text{ real}\}$ is the flow of X with fixed point p, then $\{dX_t(p) \mid t \text{ real}\}$ forms a one-parameter group of transformations of $T_p(M)$. The properties of the flow are determined near p by the infinitesimal generator of this group.

If one drops the assumption that p is a fixed point, the problem of identifying $T_p(M)$ with $T_{X_t(p)}(M)$ arises. One possibility is to provide M with a Riemannian structure and to identify $T_{X_t(p)}(M)$ and $T_p(M)$ via parallel translation along the flow. Unfortunately, the resulting set of transformations of $T_p(M)$ need not form a group. Conditions under which these transformations form a group are investigated.

The entire situation becomes clearer from the point of view of the linear frame bundle, $L(M)$. If $\tilde{X}$ is the natural lift of X to $L(M)$ and ω is a metric

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connection form on $L(M)$, then the various notions studied are related to the map $u \to \omega(\widetilde{X}(u))$: $L(M) \to gl(n)$. For instance, the set of transformations forms a one-parameter group if and only if this map is constant on orbits of $\widetilde{X}$. It seems natural to investigate the situation when the map $u \to \omega(\widetilde{X}(u))$ is of constant canonical form. This paper is devoted to the class of complete vector fields determined by these two properties. If ω is the connection form, such vector fields will be called ω-linear.

If X is ω-linear, the linear frame bundle can be reduced to a bundle on which the map $u \to \omega(\widetilde{X}(u))$ is a constant function. This reduction induces a splitting of the tangent bundles into three subbundles, an expanding, a contracting, and a third field, which separate vectors according to their behavior under the flow $\{dX_t \mid t \text{ real}\}$. Much of the dynamics of the flow $\{X_t \mid t \text{ real}\}$ is shown to depend upon the nature of this splitting.

Among the results obtained are the following. If X is ω-linear and equicontinuous on a compact manifold, the derivative flow is uniformly bounded. This fact implies that the enveloping semigroup is a Lie group and acts as a Lie transformation group on the manifold. It is shown that the ergodic and weak mixing properties of some geodesic flows follow from this approach. The minimality of certain nilflows can also be shown by use of ω-linear vector fields.

In §1 we set up the notation and review some facts to be used.

In §2 we investigate the conditions under which the set of transformations, $H\hat{X}_t \cdot \hat{X}_t$: $T_x(M) \to T_x(M)$, forms a group. We then define ω-linearity and develop the induced reduction of the frame bundles and the induced splitting of the tangent bundle. It should be noted that the splitting does not require this reduction of $L(M)$, but is facilitated by it.

In §3, ω-linearity is combined with various other hypotheses to get results involving the notions of asymptoticity, regional proximality, equicontinuity, ergodicity, minimality, and weak mixing.

In §4 these results are applied to such examples as systems of differential equations on $\mathbb{R}^n$, Lie algebra flows on Lie groups, the three dimensional nilflow, and the geodesic flows.

Finally, I would like to thank Professor R. Ellis for many stimulating conversations concerning this work.

1. Notation and preliminaries. I will use [3] as a general reference for the dynamics and [8] for the bundle theory and geometry. $L(M)$, $O(M)$, $T(M)$ and $UT(M)$ will denote the bundles of linear frames, orthonormal frames, tangent vectors, and unit tangent vectors respectively, and Π_L, Π_O, Π_T and Π_{UT} will denote the corresponding projections to M. $T(M)$ will be viewed as the bundle associated with $L(M)$ with standard fibre $\mathbf{R}^n$. Thus a typical element of $T(M)$ will be written $u \cdot \zeta$, where "$\cdot$" is formal matrix multiplication, $u \in L(M)$ is viewed as a $1 \times n$ matrix of tangent vectors, and $\zeta \in \mathbf{R}^n$ is viewed as the $n \times 1$ matrix of real coefficients. The identity $(u \cdot g) \cdot (g^{-1} \cdot \zeta) = u \cdot \zeta$ for all g in $Gl(n)$ is clear.

We will assume throughout that M is an n-dimensional connected $\mathcal{C}^\infty$ Riemannian manifold, that all maps are differentiable of class $\mathcal{C}^\infty$, and that all connections are metric connections unless specified otherwise. Connections will be specified by their connection form. We will use $\mathfrak{X}(U)$ for the set of differentiable vector fields on U, $\mathfrak{C}(M)$ for the complete vector fields on M, and (M, X) for the transformation group generated by $X \in \mathfrak{C}(M)$. The distance function will be denoted by $\delta(\ ,\ )$, the norm in $T(M)$ by $\|\ \ \|$, and the norm of a map by $|||\ \ |||$. If γ is a curve in M, the tangent vector at $\gamma(s)$ will be denoted $(d\gamma(t)/dt)_{t=s}$ or $\dot\gamma(s)$. The space derivative will be denoted by d.

A curve γ is an integral curve of $X \in \mathfrak{X}(U)$ if and only if $X(\gamma(t)) = \dot\gamma(t)$ for each t in the domain of γ. In this case, if $\gamma(0) = x$, we use the notation $\gamma(t) = X_t(x)$.

The group of nonsingular real $n \times n$ matrices, $Gl(n)$, has Lie algebra $gl(n)$, the algebra of all real $n \times n$ matrices. We will view $T_g(Gl(n))$ as the set $\{B_g \mid B \in gl(n)\}$ when viewing $gl(n)$ as a set of real matrices and as $\{B(g) \mid B \in gl(n)\}$ when viewing $gl(n)$ as the left invariant vector fields. The relationship between the two interpretations is given by

$$B(g) = (d/dt)(g \cdot e^{Bt})_{t=0} = (gB)_g.$$

For $X \in \mathfrak{X}(M)$, a *lift* of X to $L(M)$ is a vector field $\tilde{X} \in \mathfrak{X}(L(M))$ such that $R_{g*}\tilde{X} = \tilde{X}$ and $d\Pi_L(u) \cdot \tilde{X}(u) = X(\Pi_L(u))$ for each $u \in L(M)$. Any lift $\tilde{X}$ of X induces an associated lift $\hat{X} \in \mathfrak{X}(T(M))$ given by $\hat{X}_t(u \cdot \zeta) = \tilde{X}_t(u) \cdot \zeta$. Clearly each $\hat{X}_t\colon T_x(M) \to T_{X_t(x)}(M)$ is linear and $\tilde{X}_t(u) = (\hat{X}_t(u \cdot e_1), \cdots, \hat{X}_t(u \cdot e_n))$.

Throughout $\tilde{X} \in \mathfrak{C}(L(M))$ will denote the natural lift of $X \in \mathfrak{C}(M)$ given by

$$\tilde{X}_t(Y_x^1, \cdots, Y_x^n) = (dX_t(x) \cdot Y_x^1, \cdots, dX_t(x) \cdot Y_x^n).$$

Clearly the horizontal part, $H\tilde{X}$, of $\tilde{X}$ is also a lift of X, and the vertical part, $V\tilde{X}$, is a lift of the zero vector field. The associated lift $H\hat{X}$ is horizontal and the flow maps $H\hat{X}_t$ effect parallel translation along X_t.

A *normal sphere*, N_x, at x in a manifold with a connection, ω, is a sphere about x in which any two points are joined by a unique geodesic in N_x whose arc length measures distance.

2. ω-linearity.

Basic Lemma. *Let $X \in \mathfrak{C}(M)$ and let ω be a metric connection such that*

$\omega\tilde{X}(u) = A$ for some $u \in L(M)$, then $V\hat{X}$ satisfies $V\hat{X}_t(u \cdot \zeta) = u \cdot e^{At} \cdot \zeta$ whenever the expression is defined. If $[\hat{X}, H\hat{X}] = 0$, then

$$\|\hat{X}_t(u \cdot \zeta)\| = \|u \cdot e^{At} \cdot \zeta\|.$$

Proof. Let $\gamma: \mathbf{R} \to L(M)$ be given by $\gamma(t) = u \cdot e^{At}$, then $\dot{\gamma}(t) = (d/ds)(u \cdot e^{As})_{s=t} = A^*(u \cdot e^{At})$. By definition, $V\tilde{X}(u) = A^*(u)$ and $V\tilde{X}(u \cdot e^{At}) = dR_{e^{At}}(u) \cdot V\tilde{X}(u) = dR_{e^{At}}(u) \cdot A^*(u) = A^*(u \cdot e^{At})$. We have $\dot{\gamma}(t) = V\tilde{X}(\gamma(t))$ for all t so that $\gamma(t) = V\tilde{X}_t(u) = u \cdot e^{At}$. Thus

$$V\hat{X}_t(u \cdot \zeta) = V\tilde{X}_t(u) \cdot \zeta = u \cdot e^{At} \cdot \zeta,$$

completing the proof of the first statement.

Since $V\hat{X} = -H\hat{X} + \hat{X}$, if $[\hat{X}, H\hat{X}] = 0$ then $V\hat{X}_t = H\hat{X}_{-t} \cdot \hat{X}_t$ and $H\hat{X}_{-t} \cdot \hat{X}_t(u \cdot \zeta) = u \cdot e^{At} \cdot \zeta$. Since ω is a metric connection,

$$|||H\hat{X}_{-t}||| = 1 \quad \text{and} \quad \|\hat{X}_t(u \cdot \zeta)\| = \|H\hat{X}_{-t} \cdot \hat{X}_t(u \cdot \zeta)\| = \|u \cdot e^{At} \cdot \zeta\|$$

for each $u \in L(M)$, $\zeta \in \mathbf{R}^n$, and $t \in \mathbf{R}$.

2.0. **Lemma.** *Let $Y \in \mathfrak{X}(L(M))$ be any lift of $X \in \mathfrak{X}(M)$ and let ω be a connection on $L(M)$, then $[Y, V\tilde{X}] = 0$ if and only if $\omega\tilde{X}(Y_t(u)) = \omega\tilde{X}(u)$, for all $u \in L(M)$ and $t \in \mathbf{R}$.*

Proof. Since Y is a lifted vector field, $dY_t(u) \cdot A^*(u) = A^*(Y_t(u))$ for each $A \in gl(n)$. We have $[Y, V\tilde{X}] = 0$ if and only if $V\tilde{X}(Y_t(u)) = dY_t(u) \cdot V\tilde{X}(u)$ for all $t \in \mathbf{R}$ and $u \in L(M)$. Since $dY_t(u) \cdot V\tilde{X}(u) = dY_t(u) \cdot (\omega\tilde{X}(u))^*(u) = (\omega\tilde{X}(u))^*(Y_t(u))$, $[Y, V\tilde{X}] = 0$ if and only if $\omega\tilde{X}(Y_t(u)) = \omega V\tilde{X}(Y_t(u)) = \omega\tilde{X}(u)$.

2.1. **Proposition.** *Let ω be a linear connection. The following statements are pairwise equivalent.*

(1) *The vector field $[\tilde{X}, H\tilde{X}] = 0$.*

(2) *The vector field $[\hat{X}, H\hat{X}] = 0$.*

(3) *The function $\omega\tilde{X}$; $L(M) \to gl(n)$ is constant on orbits of $\tilde{X}$.*

(4) *The function $\omega\tilde{X}$ is constant on orbits of $H\tilde{X}$.*

(5) *For each $t \in \mathbf{R}$, the function (X_{t*}): $\mathfrak{X}(M) \to \mathfrak{X}(M)$ commutes with the function ∇_X: $\mathfrak{X}(M) \to \mathfrak{X}(M)$.*

The equivalence of (1) and (2) is immediate from the definition of $\tilde{X}$.

Since $0 = [\tilde{X}, \tilde{X}] = [\tilde{X}, H\tilde{X}] + [\tilde{X}, V\tilde{X}]$, we have $[\tilde{X}, H\tilde{X}] = 0$ if and only if $[\tilde{X}, V\tilde{X}] = 0$. Setting $Y = \tilde{X}$ in 2.0, we see that (1) is equivalent to (3). Also since $[\tilde{X}, V\tilde{X}] = [H\tilde{X}, V\tilde{X}] + [H\tilde{X}, H\tilde{X}]$, $[\tilde{X}, H\tilde{X}] = 0$ if and only if $[H\tilde{X}, V\tilde{X}] = 0$. Setting $Y = H\tilde{X}$ in 2.0, we see that (1) is equivalent to (4).

The equivalence of (2) and (5) follows immediately from the characterization

$$(\nabla_X Y)(x) = \lim_{t\to 0} \frac{1}{t} [H\hat{X}_t(Y(X_t(x))) - Y_{(x)}]$$

and the fact that $[\tilde{X}, H\tilde{X}] = 0$ if and only if $\hat{X}_s \cdot H\hat{X}_t = H\hat{X}_t \cdot \hat{X}_s$ for all t and s in **R**.

2.2. **Definition.** A vector field X in $\mathfrak{X}(M)$ will be called *ω-linear* if there is a metric connection ω such that the function $\omega\tilde{X}$: $L(M) \to gl(n)$ is constant on orbits of $\tilde{X}$ and of constant canonical form on all of $L(M)$.

2.3. **Theorem.** *Let X be ω-linear. For each $u_0 \in L(M)$, $R(M) = \{u \in L(M) \mid \omega\tilde{X}(u) = \omega\tilde{X}(u_0)\}$ is a reduction of $L(M)$ with group $\mathcal{C}(\omega\tilde{X}(u_0)) = \{g \in Gl(n) \mid g\omega\tilde{X}(u_0)g^{-1} = \omega\tilde{X}(u_0)\}$.*

Proof. Let $\omega\tilde{X}(u_0) = A$. For each $u \in R(M)$, $u \cdot g \in R(M)$ if and only if $g \in \mathcal{C}(A)$, since $\omega\tilde{X}(ug) = g^{-1}\omega X(u)g = g^{-1}Ag$. Thus each fibre intersects $R(M)$ in exactly one orbit of $\mathcal{C}(A)$. It remains to be shown that there are local cross sections of $L(M)$ which map into $R(M)$.

Let f: $Gl(n) \to gl(n)$ be the map $f(g) = gAg^{-1}$. For each $g \in Gl(n)$, we express $T_g(Gl(n))$ as $\{B(g) \mid B \in gl(n)\}$; then

$$\begin{aligned} df(g) \cdot B(g) &= (d/dt)(f(g \cdot e^{Bt}))_{t=0} = (d/dt)(g \cdot e^{Bt} \cdot A \cdot e^{-Bt} \cdot g^{-1})_{t=0} \\ &= (d/dt)(g \cdot e^{Bt} \cdot A)_{t=0} \cdot (e^{-Bt} \cdot g^{-1})_{t=0} \\ &\quad + (g \cdot e^{Bt} \cdot A)_{t=0} \cdot (d/dt)(e^{-Bt} \cdot g)_{t=0} \\ &= gBAg^{-1} - gABg^{-1} = g(BA - AB)g^{-1}. \end{aligned}$$

Clearly $df(g) \cdot B(g) = 0$ if and only if $[B, A] = BA - AB = 0$ and $\dim(\ker(df(g))) = \dim(\ker[\cdot, A])$. Thus df is of constant rank k, and by the rank theorem on manifolds there are coordinate patches (W, ϕ) about I in $gl(n)$ and (V, ψ) about A in $gl(n)$ such that

$$\psi \cdot f \cdot \phi^{-1}(x_1, \cdots, x_{n^2}) = (x_1, \cdots, x_k, 0, \ldots)$$

on all of $\phi(W)$.

Let $x \in M$ and let σ: $U \to L(M)$ be a local cross section about x such that $\omega\tilde{X}(\sigma(x)) = A$. For each $y \in U$, denote $\phi^{-1} \cdot \psi \cdot \omega\tilde{X} \cdot \sigma(y)$ by g_y and define a local cross section ρ about x by $\rho(y) = \sigma(y) \cdot g_y$. Since $f \cdot \phi^{-1} \cdot \psi|_{\mathrm{Im}(f)}$ is the identity, $g_y A g_y^{-1} = f(g_y) = f(\phi^{-1} \cdot \psi \cdot \omega\tilde{X} \cdot \sigma(y)) = \omega\tilde{X}\sigma(y)$ so that

$$A = g_y^{-1}\omega\tilde{X}(\sigma(y))g_y = \omega\tilde{X}(\sigma(y)g_y) = \omega\tilde{X}(\rho(y)).$$

Thus ρ is a cross section about x into $R(M)$ and the result follows.

2.4. **Corollary.** *If X is ω-linear and if $\overline{\{X_t(\Pi_L(u_0)) \mid t \text{ real}\}} = M$ for some $u_0 \in L(M)$, $S(M) = \{u \in O(M) \mid \omega\tilde{X}(u) = \omega\tilde{X}(u_0)\}$ is a reduction of $O(M)$ with group $\mathcal{C}(\omega\tilde{X}(u_0)) \cap O(n)$.*

Proof. We show that each fibre meets $S(M)$ in exactly one $(\mathcal{C}(A) \cap O(n))$-orbit and that there exist local cross sections into $S(M)$. Let $\omega\tilde{X}(u_0) = A$ and suppose there is $x \in M$ such that $\Pi_0^{-1}(x) \cap S(M) = \emptyset$, then choose a compact neighborhood U of x and a sequence $\{t_n\}_1^{+\infty}$ of real numbers such that $X_{t_n}(y) \to x$ as $n \to +\infty$ with $X_{t_n}(y) \in U$ for each n. The sequence $\{u_n\}$ with $u_n = \tilde{HX}_{t_n}(u_0)$ in the compact set $\Pi_0^{-1}(U)$ has a convergent subsequence $u_{n_i} \to u \in \Pi_0^{-1}(x)$. But then $\omega\tilde{X}(u) = \lim \omega\tilde{X}(u_{n_i}) = A$. This is a contradiction and thus each fibre meets $S(M)$. The rest of the proof is exactly the same as the proof of the theorem.

The corollary provides a means of reducing the group of the bundle to a finite group if the matrix A is sufficiently nontrivial. If, for instance, M is simply connected and such a vector field X exists on M, the group is reduced to the identity. We know, therefore, that no point-transitive vector field of this type can exist on an odd-dimensional sphere.

Let $A \in gl(n)$ and let $\mathcal{P}(x)$ be the minimal polynomial of A. Let $\mathcal{P}^+(x)$, $\mathcal{P}^-(x)$, and $\mathcal{P}^0(x)$ be polynomial factors of $\mathcal{P}(x)$ corresponding to the eigenvalues with positive, negative, and zero real parts respectively and let $\mathcal{P}^1(x)$ be the product of all unique factors in $\mathcal{P}^0(x)$. Finally, let V^+, V^-, V^0 and V^1 be the null spaces of the respective polynomials. Thus $\mathbf{R}^n = V^+ \oplus V^- \oplus V^0$ and $V^1 \leq V^0$. We note that each $V^{(\ )}$ is $\mathcal{C}(A)$-invariant since for $\zeta \in V^{(\ )}$ and $g \in \mathcal{C}(A)$, $\mathcal{P}^{(\ )}(A)g\zeta = g\mathcal{P}^{(\ )}(A)\zeta = g \cdot 0 = 0$. Thus $g\zeta \in V^{(\ )}$ and $gV^{(\ )} = V^{(\ )}$.

Let X be ω-linear with $\omega\tilde{X}(u_0) = A$ and let $R(M)$ be the corresponding reduction of $L(M)$. Define distributions T^+, T^-, T^0 and T^1 on M by $T^{(\ )}(x) = u \cdot V^{(\ )} = \{u \cdot \zeta | \zeta \in V^{(\ )}\}$ for some $u \in \Pi_R^{-1}(x)$. If $v \in \Pi_R^{-1}(x)$ also, then $v = ug$ for some g in $\mathcal{C}(A)$ and $v \cdot V^{(\ )} = ug \cdot V^{(\ )} = u \cdot gV^{(\ )} = u \cdot V^{(\ )}$ since $V^{(\ )}$ is $\mathcal{C}(A)$-invariant. Thus $T^{(\ )}$ is well defined.

2.5. **Theorem.** *Let X be ω-linear.*

(1) *For each x in M, $T_x(M) = T^+(x) \oplus T^-(x) \oplus T^0(x)$.*

(2) *T^+ is expanding in the sense that, for each $u \cdot \zeta$ in $T^+(x)$, $\|\hat{X}_t(u \cdot \zeta)\| \to +\infty$ as $t \to +\infty$ and $|||\hat{X}_t|_{T^+(x)}||| \to 0$ uniformly on M as $t \to -\infty$.*

(3) *T^- is contracting in the sense that, for $u \cdot \zeta$ in $T^-(x)$, $\|\hat{X}_t(u \cdot \zeta)\| \to +\infty$ as $t \to -\infty$ and $|||\hat{X}_t|_{T^-(x)}||| \to 0$ uniformly on M as $t \to +\infty$.*

(4) *$|||X_t|_{T^1(x)}|||$ is uniformly bounded for all $t \in \mathbf{R}$ and $x \in M$.*

Proof. The first statement follows immediately from the fact that $\mathbf{R}^n = V^+ \oplus V^- \oplus V^0$.

Let $u \cdot \zeta \in T^+(M)$ with $u \in O(M)$; then $\|u \cdot \zeta\| = (\Sigma_{i=1}^n \zeta_i^2)^{1/2} = \|\zeta\|$ and $\|\hat{X}_t(u \cdot \zeta)\| = \|V\hat{X}_t(u \cdot \zeta)\| = \|ue^{At} \cdot \zeta\| = \|u \cdot e^{At}\zeta\| = \|e^{At}\zeta\|$. Let A be in rational canonical form

$$\begin{pmatrix} A_+ & & 0 \\ & A_- & \\ 0 & & A_0 \end{pmatrix},$$

where $A_+ = A|_{V^+}$, $A_- = A|_{V^-}$ and $A_0 = A|_{V^0}$. Let K be a complex matrix such that $K^{-1}A_+K$ is in Jordan form

$$\begin{pmatrix} \begin{pmatrix} \lambda_1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_1 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \lambda_q & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_q \end{pmatrix} \end{pmatrix}$$

then $e^{K^{-1}A_+Kt} = K^{-1}e^{A_+t}K$ has the form

$$\begin{pmatrix} e^{\lambda_1 t}\begin{pmatrix} 1 & t & \cdots & t^{s_1}/s_1! \\ & \ddots & & \vdots \\ & & \ddots & t \\ 0 & & & 1 \end{pmatrix} & & \\ & \ddots & \\ & & e^{\lambda_q t}\begin{pmatrix} 1 & t & \cdots & t^{s_q}/s_q! \\ & \ddots & & \vdots \\ & & \ddots & t \\ 0 & & & 1 \end{pmatrix} \end{pmatrix}$$

Since $\mathrm{Re}(\lambda_1), \cdots, \mathrm{Re}(\lambda_q)$ are all positive numbers, the modulus of each entry of $K^{-1}e^{A_+t}K$ tends exponentially to zero as $t \to -\infty$ and tends to $+\infty$ as $t \to +\infty$. Therefore each entry of the real matrix e^{A_+t} tends to zero as $t \to -\infty$ and tends to $+\infty$ as $t \to +\infty$. Thus $\|e^{A_+t}\| \to 0$ as $t \to -\infty$ and $\|e^{A_+t}\zeta\| \to +\infty$ as $t \to +\infty$ for each $\zeta \in V^+$. Thus $\|\hat{X}_t(u \cdot \zeta)\| = \|e^{A_+t} \cdot \zeta\| \to +\infty$ as $t \to +\infty$ for each $\zeta \in V^+$. Since $\|\hat{X}_t(u \cdot \zeta)\|/\|u \cdot \zeta\| = \|e^{A_+t}\zeta\|/\|\zeta\|$,

$$|||\hat{X}_t|_{T^-(x)}||| = |||e^{At}|_{V^+}||| = |||e^{A_+t}||| \longrightarrow 0$$

independent of $x \in M$ as $t \longrightarrow +\infty$. This completes the proof of (2). The proof of (3) is exactly analogous.

Finally, the map $A_1 = A|_{V1}$ has canonical form

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -b_1^2 & 0 \end{pmatrix} & & & & & \\ & \ddots & & & & \\ & & \begin{pmatrix} 0 & 1 \\ -b_q^2 & 0 \end{pmatrix} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

and

$$e^{A_1t} = \begin{pmatrix} \begin{pmatrix} \cos(b_1t)\ 1/b_1 & \sin(b_1t) \\ -1/b_1 \quad \sin(b_1t) & \cos(b_1t) \end{pmatrix} & & & & & \\ & \ddots & & & & \\ & & \begin{pmatrix} \cos b_qt\ 1/b_q & \sin(b_qt) \\ -1/b_q \quad \sin(b_qt) & \cos(b_qt) \end{pmatrix} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Each entry is uniformly bounded above by the number $\sup_{i=1,\dots,q}\{|b_i|, |1/b_i|, 1\}$. Thus $|||e^{A_1t}|||$ is uniformly bounded for all $t \in \mathbf{R}$. Since $\|X_t(u \cdot \zeta)\|/\|u \cdot \zeta\| = \|e^{At}\zeta\|/\|\zeta\|$, $|||\hat{X}_t|_{T1(x)}|||$ is equal to $|||e^{A_1t}|||$ for every $x \in M$ and therefore $|||\hat{X}_t|_{T1(x)}|||$ is uniformly bounded for all t in $\mathbf{R}$ and $x \in M$.

3. Dynamical results. Before proceeding with the dynamical results, we review the definitions of the notions involved.

3.1. Definitions. Let (M, X) be a transformation group and let x, y be points of M.

(1) The point x is *asymptotic* to y if to each $\epsilon > 0$ there corresponds an $N > 0$ such that $\delta(X_t(x), X_t(y)) < \epsilon$ for all $t > N$ (positively asymptotic) or $\delta(X_t(x), X_t(y)) < \epsilon$ for all $t < -N$ (negatively asymptotic).

(2) The point x is *proximal* to y if to each $\epsilon > 0$ there corresponds a t in $\mathbf{R}$ such that $\delta(X_t(x), X_t(y)) < \epsilon$.

(3) The point x is *regionally proximal* to y if, to each $\epsilon > 0$ and each pair of neighborhoods U of x and V of y, there correspond points a in U, b in V and t in $\mathbf{R}$ such that $\delta(X_t(a), X_t(b)) < \epsilon$. The set of regionally proximal pairs will be denoted by $Q(M, X)$. The smallest closed X-invariant equivalence relation containing $Q(M, X)$ is called the *equicontinuous structure relation* and is denoted $S(M, X)$.

(4) (M, X) is *distal* if the only pairs of proximal points are those on the diagonal $\Delta = \{(x, x) | x \in M\}$.

(5) An *extensive* set of real numbers is a bisequence $\{t_n\}_{-\infty}^{\infty}$ such that $\lim_{n\to+\infty} t_n = +\infty$ and $\lim_{n\to+\infty} t_n = -\infty$.

(6) A point x is *recurrent* if there is an extensive set $\{t_n\}$ such that $\lim_{n\to+\infty} X_{t_n}(x) = x$ and $\lim_{n\to-\infty} X_{t_n}(x) = x$.

(7) (M, X) is *equicontinuous* if to each $\epsilon > 0$ there is an $\eta > 0$ such that $\delta(X_t(x), X_t(y)) < \epsilon$ for all t in $\mathbf{R}$ and all x, y in M such that $\delta(x, y) < \eta$.

(8) (M, X) is *topologically ergodic* if there are no proper X-invariant subsets $C = \overline{C^0}$ of M. (This is clearly equivalent to the definition given in [7].)

(9) (M, X) is *topologically weakly mixing* if the diagonal flow $(M \times M, X)$ is ergodic.

(10) (M, X) is *minimal* if there are no proper closed X-invariant subsets of M.

(11) The *enveloping semigroup* of (M, X), denoted $E(M, X)$, is the closure of the flow $\{X_t | t \text{ real}\}$ in M^M with respect to the topology of pointwise convergence. The semigroup operation is composition of functions.

(12) Let X be ω-linear with $\omega\tilde{X}(u_0) = A$ and let $\sigma\colon U \to \Pi_R^{-1}(U)$ be a local cross section into the corresponding reduced bundle of frames, $R(M)$. We will call Y in $\mathfrak{X}(U)$ a *special zero* vector field on U if there is a block in A of the form

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

whose first row is coincident with row k of A, and $Y(y) = \sigma(y) \cdot e_k$ for all y in U. The definition says that e_k is a zero eigenvector of A with multiplicity or that $e_k \in \ker A \cap \operatorname{Im} A$.

The corresponding vector fields for the pure imaginary case will be called special imaginary vector fields. That is Y in $\mathfrak{X}(U)$ will be called *special imaginary* on U if there is a block in A of the form

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -b^2 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & & 0 \\ & \begin{pmatrix} 0 & 1 \\ -b^2 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \\ & & \ddots & \\ 0 & & & \begin{pmatrix} 0 & 1 \\ -b^2 & 0 \end{pmatrix} \end{pmatrix}$$

whose first row is coincident with row k of A and $Y(y) = \sigma(y) \cdot e_k$ for all y in U.

(13) A vector field Y in $\mathfrak{X}(U)$ will be called a *T^+-vector field, T^--vector field* or *T^0-vector field* if $Y(y) \in T^+(y)$, $Y(y) \in T^-(y)$ or $Y(y) \in T^0(y)$ $(y \in U)$ respectively.

(14) A map $f\colon (M, X) \to (N, Y)$ is called a *homomorphism* if $f(X_t(x)) = Y_t(f(x))$ for each x in M and t in $\mathbf{R}$.

3.2. **Lemma.** *If Y is special zero or special imaginary, then $(x, Y_s(x)) \in Q(M, X)$ for all $x \in M$ and $s \in R$ such that $Y_s(x)$ is defined.*

Proof. If $Y \in \mathfrak{X}(U)$ is a special zero vector field, define $Y^a \in \mathfrak{X}(U)$ by $Y^a(y) = \sigma(y)(e_k + ae_{k+1})$. Let $B = \sup_{Y \in U; i=1,\cdots,n}\{\|\sigma(y) \cdot e_i\|\}$ and let $x \in M$ and $s \in \mathbf{R}$ be chosen such that $Y_s(x)$ is defined. Given $\epsilon > 0$ and an open set V containing $Y_s(x)$, pick a between zero and ϵ/Bs such that $Y_s^a(x) \in V$. This is possible since $Y^a \to Y^0 = Y$ as $a \to 0$. We have

$$\begin{aligned} \|\hat{X}_t Y^a(y)\| &= \|\hat{X}_t((y) \cdot (e_k + ae_{k+1}))\| \\ &= \|\sigma(y) \cdot e^{At}(e_k + ae_{k+1})\| \\ &= \|\sigma(y) \cdot ((1 + at)e_k + ae_{k+1})\| \end{aligned}$$

for each y in U, since

$$\exp\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} t = \begin{pmatrix} 1 & t & \cdots & t^q/q! \\ & \ddots & \ddots & \vdots \\ & & \ddots & t \\ 0 & & & 1 \end{pmatrix}.$$

Thus $\|\hat{X}_{-1/a}(Y^a(y))\| = \|\sigma(y) \cdot ae_{k+1}\| = a\|\sigma(y) \cdot e_{k+1}\| \le aB$. The curve $r \to X_{-1/a}(Y_r^a(x))$ runs from $X_{-1/a}(x)$ at $r = 0$ to $X_{-1/a}(Y_s^a(x))$ at $r = s$ with length

$$\int_0^s \|d/dr(X_{-1/a}(Y_r^a(x)))\|\, dr = \int_0^s \|\hat{X}_{-1/a}(Y^a(Y_r^a(x)))\|\, dr$$
$$= \int_0^s aB\ dr = aBs < \epsilon.$$

This implies $\delta(X_{-1/a}(x), X_{-1/a}(Y_s^a(x))) < \epsilon$ So, given $\epsilon > 0$ and neighborhoods V of $Y_s(x)$ and W of x there are points $Y_s^a(x)$ in V, x in W and $-1/a$ in $\mathbf{R}$ such that $\delta(X_{-1/a}(x), X_{-1/a}(Y_s^a(x))) < \epsilon$. Thus $(x, Y_s(x)) \in Q(M, X)$.

If Y is special imaginary, define Y^a in $\mathfrak{X}(U)$ by $Y^a(y) = \sigma(y) \cdot (e_k + ae_{k+3})$ for all y in U. Let $x \in M$ and $s \in \mathbf{R}$ such that $Y_s(x)$ is defined. Given $\epsilon > 0$ and V containing $Y_s(x)$, pick an integer m such that $a = b^3/\pi m$ is between 0 and ϵ/Bs and such that $Y_s^a(x) \in V$. We have

$$\|\hat{X}_t(Y^a(y))\| = \|\sigma(y) \cdot e^{At}(e_k + ae_{k+3})\|$$
$$= \|\sigma(y) \cdot [(\cos bt(1 - at/2b^2) + a/2b^3 \sin bt)e_k$$
$$+ ((at/2b - b) \sin bt)e_{k+1}$$
$$+ (a/b \sin bt)e_{k+2} + (a \cos bt)e_{k+3}]\|$$

since

$$\exp\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -b^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \\ & \ddots \\ & \begin{pmatrix} 0 & 1 \\ -b^2 & 0 \end{pmatrix} \end{pmatrix} t$$

$$= \begin{pmatrix} \cos bt & 1/b \sin bt & t/2b \sin bt & 1/2b^3(\sin bt - bt \cos bt) \\ -b \sin bt & \cos bt & 1/2b(\sin bt + bt \cos bt) & t/2b(\sin bt) \\ 0 & 0 & \cos bt & 1/b \sin bt \\ 0 & 0 & -b \sin bt & \cos bt \end{pmatrix}$$

Thus $\|\hat{X}_{2\pi m/b}(Y^a(y))\| = \|\sigma(y) \cdot ae_{k+3}\| = a\|\sigma(y) \cdot e_{k+3}\| \leq aB$. As above we have

$$\delta(X_{2\pi m/b}(x), X_{2\pi m/b}(Y_s^a(x))) \leq \int_0^s \|\hat{X}_{2\pi m/b}(Y^a(Y_r^a(x)))\| dr$$
$$= \int_0^s aB\, dr = aBs < \epsilon$$

and thus $(x, Y_s(x)) \in Q(M, X)$.

In proving 3.2 we established the following corollary which will be useful later.

3.3. Corollary. *Let $Y \in \mathfrak{X}(U)$ be a special zero vector field and let $s \in \mathbf{R}$ and $x \in M$ such that $Y_s(x)$ is defined. There exist nearby vector fields, $\{Y^a \in \mathfrak{X}(U) | 0 < a \leq 1\}$, such that $\lim_{a\to 0} Y_s^a(x) = Y_s(x)$ and $\lim_{a \to 0} \delta(X_{-1/a}(x), X_{-1/a}(Y_s^a(x))) = 0$.*

3.4. Lemma. *If $Y \in \mathfrak{X}(U)$ is a T^+-vector field, then x and $Y_s(x)$ are negatively asymptotic whenever defined. If Y is T^-, x and $Y_s(x)$ are positively asymptotic.*

Proof. The distance $\delta(X_t(x), X_t(Y_s(x)))$ is less than or equal to the arc length $\int_0^s \|d/dr(X_t(Y_r(x)))\| dr = \int_0^s \|\hat{X}_t \cdot Y(Y_r(x))\| dr$ of the curve $r \to X_t(Y_r(x))$. If Y is T^+, $\|\hat{X}_t \cdot Y(Y_r(x))\| \to 0$ as $t \to -\infty$. Thus $\delta(X_t(x), X_t(Y_s(x))) \to 0$ as $t \to -\infty$. If Y is T^-, $\|\hat{X}_t \cdot Y(Y_r(x))\| \to 0$ as $t \to +\infty$. Thus $\delta(X_t(x), X_t(Y_s(x))) \to 0$ as $t \to +\infty$.

3.5. Theorem. *Let M be compact and let $u_0 \in L(M)$. If X is ω-linear and $\omega\tilde{X}(u_0) = A$, the following are equivalent.*

(1) *The matrix A is diagonalizable over the complex numbers with only zero and pure imaginary eigenvalues.*

(2) *(M, X) has uniformly bounded derivatives.*

(3) *(M, X) is equicontinuous.*

Proof. If $A = A_0$ is complex diagonalizable, then the minimal polynomial consists of unique irreducible factors. Thus $V^1 = R^n$ and $T^1(x) = T_x(M)$ for all x in M. By (3) of Theorem 2.5, $|||\hat{X}_t|_{T^1(x)}||| = |||\hat{X}_t||| \leq B$ for all t in R and x in M, so (1) implies (2).

For each x in M, let $N(x, r_x)$ be a normal sphere of radius r_x about x and let S_x be a sphere of radius $r_x/2$ about x. Since M is compact, there is a finite cover $\{S_{x_i} | i = 1, \cdots, m\}$ of M. Let $R = \inf_{i=1,\cdots,m} r_{x_i}/2$. Given $\epsilon > 0$ choose η the smaller of ϵ/B and R, then if $\delta(x, y) < \eta$ and $x \in S_{x_i}$, x and y are in $N(x_i, r_{x_i})$. Let γ be the unique geodesic in $N(x_i, r_{x_i})$ with $\gamma(0) = x$ and $\gamma(s) = y$, then

$$\delta(X_t(x), X_t(y)) \leq \int_0^s \|d/dr(X_t(\gamma(r)))\| dr = \int_0^s \|\hat{X}_t(\dot{\gamma}(r))\| dr \leq \int_0^s B\|\dot{\gamma}(r)\| dr$$
$$\leq B\delta(x, y) \leq B\eta = \epsilon.$$

Thus (2) implies (3).

If A has eigenvalues with nonzero real parts, T^+ or T^- must be nontrivial. Thus there is a pair of asymptotic points. Asymptotic points are regionally proximal and thus (M, X) is not equicontinuous by a similar argument to that found in [3, p. 25].

If A has only zero or pure imaginary eigenvalues, but A is not complex diagonalizable, from 3.2 we see that there are regionally proximal points. Again (M, X) is not equicontinuous. Thus (3) implies (1) and the result is established.

We note that the only use of compactness was in establishing the existence of a number R such that if $\delta(x, y) < R$, then there is a geodesic (minimizing) connecting x and y whose arc length is $\delta(x, y)$. If M is noncompact and has this property, the theorem holds on M. The manifold $\mathbf{R}^n$ is an example.

With this characterization of equicontinuity it is possible to prove that the enveloping semigroup acts as a Lie transformation group on M.

3.6. **Theorem.** *If X is ω-linear on a compact manifold M such that (M, X) is equicontinuous, then $(M, E(M, X))$ is a Lie transformation group.*

Proof. It is shown in [3, pp. 25, 26], that $E(M, X)$ forms a compact topological transformation group of homeomorphisms of M whenever (M, X) is equicontinuous. It is shown in [10, Chapter 5] that such a transformation group forms a Lie transformation group if each transformation is a differentiable map.

Let $f \in E(M, X)$ and let $x \in M$. Choose compact coordinate charts (U, ϕ) and (V, ψ) at x and $f(x)$ respectively such that $f(U) \subseteq V$, and choose ϵ to be any small positive number such that $V_1 = \{x \in V | \delta(x, V') > \epsilon\}$ is a nonempty open set containing $f(x)$.

Let $\{X_{t_k}\}_{k=1}^{\infty}$ be a sequence such that $X_{t_k} \to f$ and $X_{t_k}(x) \in V_1$ for each k. Since $\{X_t \mid t \text{ real}\}$ is equicontinuous there is an $\eta > 0$ such that $\delta(X_t(x), X_t(y)) < \epsilon$ whenever $\delta(x, y) < \eta$. Thus, if $y \in S(x, \eta) = \{z \in M | \delta(z, x) < \eta\}$, then $X_{t_k}(y) \in S(X_{t_k}(x), \epsilon) \subseteq V$. Choose an interval $[-a, a] = I$ such that $I^n \subseteq \phi(S(x, \eta))$, then $\psi \cdot X_{t_k} \cdot \phi^{-1}$ maps I^n into $\psi(V)$ for each k. Since $T(U)$ is a metric space, since $\{\hat{X}_{t_k}\}_{k=1}^{\infty}$ is an equicontinuous sequence and since $\{\hat{X}_{t_k}(u \cdot \zeta) | k = 1, \cdots, \infty\}$ is precompact for each $u \cdot \zeta \in T(M)$, it follows by the Ascoli theorem that there is a subsequence $\{\hat{X}_{t_{k_l}}\}$ and a continuous function $g\colon T(U) \to T(V)$ such that $\hat{X}_{t_{k_l}} \to g$ uniformly on $\phi^{-1}(I^n)$.

Define $S_l\colon I \to \psi(V)$ by $S_l(r) = \psi \circ X_{t_{k_l}} \circ \phi^{-1}(v + re_i)$, where v is some fixed vector in I^n. Then $S_l'(r) = d(S_l(r))/dr = (d/dr)(\psi \circ X_{t_{k_l}} \circ \phi^{-1}(v + re_i)) \cdot d\psi \circ \hat{X}_{t_{k_l}} \circ d\phi^{-1}(\partial/\partial x_i(v + re_i))$ and S_l' converges uniformly to $d\psi \circ g \circ d\phi^{-1}(\partial/\partial x_i(v + re_i))$. Since $S_l(0)$ also converges to $\psi \circ f \circ \phi^{-1}(v)$,

$$\lim_{l \to \infty} d/dr(S_l(r))_{r=0} = d/dr\left(\lim_{l \to \infty} S_l(r)\right)_{r=0}$$

or

$$d\psi \circ g d\phi^{-1}(\partial/\partial x_i(v)) = d/dr(\psi \circ f \circ \phi^{-1}(v + re_i))_{r=0} = \partial(\psi \circ f \circ \phi^{-1})/\partial x_i(v).$$

Thus the partial derivatives of $\psi \circ f \circ \phi$ exist and are continuous since g is continuous. The result now follows.

3.7. **Lemma.** *Let X be ω-linear and let $\sigma: U \to R(M)$ be a cross section about $x \in M$. For each $\zeta \in \mathbf{R}^n$, we define $Y^\zeta \in \mathfrak{X}(U)$ by $Y^\zeta(y) = \sigma(y) \cdot \zeta$ for $y \in U$. The map $f: \mathbf{R}^n = V^+ \oplus V^- \oplus V^0 \to M$ given by*

$$f(\zeta^+, \zeta^-, \zeta^0) = Y_1^{\zeta^+} \circ Y_1^{\zeta^-} \circ Y_1^{\zeta^0}(x)$$

is a diffeomorphism of a neighborhood of the origin onto a neighborhood of x.

3.8. **Lemma.** *Let X be ω-linear and let C be a closed X-invariant subset of M. If x is a recurrent point in C' and $Y \in \mathfrak{X}(U)$ is a T^+ or T^- vector field, then $Y_s(x) \in C'$ whenever defined.*

Proof. Let Y be T^+ and let $\{t_n\}$ be an extensive set such that $\lim_{n \to -\infty} X_{t_n}(x) = x$. By 3.4, $\lim_{n \to -\infty} \delta(X_{t_n}(x), X_{t_n}(Y_s(x))) = 0$ so that $\lim X_{t_n}(Y_s(x)) = \lim X_{t_n}(x) = x \in C'$. Since C' is an open X-invariant set, $X_{t_{n_0}}(Y_s(x)) \in C'$ for some n_0 and thus $Y_s(x) \in C'$.

In the next theorem we consider conditions under which a flow is ergodic. The corollary that follows generalizes the hypothesis so as to apply to the geodesic flow on a compact manifold of constant negative curvature. This example is examined in the last section.

3.9. **Theorem.** *Let X be ω-linear such that $T^0(x)$ is generated by $X(x)$ for each x in M. If (M, X) is recurrent on a dense set of points, (M, X) is ergodic.*

Suppose, to the contrary, that there is a proper X-invariant subset $C = \overline{C^0}$ of M. Choose p on the boundary of C. By 3.7, points of the form $Y_1^+ \circ Y_1^- \circ Y_1^0(p)$ cover a neighborhood U of p where Y^+, Y^- and Y^0 are T^+, T^- and T^0 respectively. Since T^0 is one dimensional the X orbits are maximal integral manifolds of T^0 and thus $Y_1^0(p)$ is in C for each T^0 vector field, Y^0.

Suppose there is a point $y = Y_1^- \circ Y_1^0(p) \in C'$. Since p and $Y_1^0(p)$ are on the boundary of C^0, $y = Y_1^-(Y_1^0(p)) \in C'$ is on the boundary of $Y_1^-(C^0)$ and thus $Y_1^-(C^0) \cap C'$ is a nonempty open set from which we can pick a recurrent point z. Since $Y_{-1}^-(z) \in C^0$, this contradicts 3.8. Thus there are no points of the form $Y_1^- \circ Y_1^0(p)$ in C'.

Let $y = Y_1^+ \circ Y_1^- \circ Y_1^0(p)$ be a recurrent point in C'. Since $Y_1^- \circ Y_1^0(p) \in C$, this also contradicts 3.8. Thus, no such set C exists and (M, X) is ergodic.

3.10. **Corollary.** *Let X be ω-linear and let $f: (M, X) \to (M, Z)$ be a homomorphism. If $df(x) \cdot (T^0(x))$ is generated by $Z(f(x))$ for each x in M and if (M, X) is recurrent on a dense set, (N, Z) is ergodic.*

Proof. Suppose to the contrary that $C = \overline{C^0}$ is a proper Z-invariant subset of

N, then $D = f^{-1}(C)$ and $\overline{D^0}$ are T^0-invariant subsets of M. Since (M, X) is recurrent on a dense set of points, this constitutes a contradiction just as in the proof of the theorem.

3.11. **Corollary.** *Let X be ω-linear and let R be the set of recurrent points of M. If U is an open X-invariant subset of $\overline{R}$, then $\overline{U}$ is T^+ and T^- invariant.*

There are similar results involving the stronger notion of minimality as shown in the next theorem. The proof is the same except for obvious modifications.

3.12. **Theorem.** *Let X be ω-linear such that $T^0(x)$ is generated by $X(x)$ for each x in M. If (M, X) is recurrent at each point of M, (M, X) is minimal.*

Under suitable conditions on M if (M, X) is equicontinuous, then orbit closures are minimal (see [3]). Thus, one might expect the orbit closures of a densely recurrent flow to be minimal whenever $T^0 = T^1$ regardless of the dimension. This is indeed the case.

3.13. **Lemma.** *If x and y are asymptotic points and y is recurrent, then the distance from the orbit of x to any point z is less than or equal to the distance from the orbit of y to z.*

Proof. Given $\epsilon > 0$, choose $t_1 \in R$ such that $\delta(X_{t_1}(y), z) < \delta(\mathrm{Orb}_X(y), z) + \epsilon/3$. Since $X_{t_1}(y)$ is recurrent, there is an extensive set $\{s_n\}$ such that, for each n, $\delta(X_{s_n}(X_{t_1}(y)), X_{t_1}(y)) < \epsilon/3$. Since x and y are asymptotic, there is an n_0 such that $\delta(X_{s_{n_0}}(X_{t_1}(x)), X_{s_{n_0}}(X_{t_1}(y))) < \epsilon/3$. Thus

$$\begin{aligned}\delta(X_{s_{n_0}}(X_{t_1}(x)), z) &\le \delta(X_{s_{n_0}}(X_{t_1}(x)), X_{s_{n_0}}(X_{t_1}(y))) \\ &\quad + \delta(X_{s_{n_0}}(X_{t_1}(y)), X_{t_1}(y)) + \delta(X_{t_1}(y), z) \\ &\le \delta(\mathrm{Orb}_X y, z) + \epsilon.\end{aligned}$$

Since ϵ was arbitrary, $\delta(\mathrm{Orb}_X x, z) \le \delta(\mathrm{Orb}_X(y), z)$.

3.14. **Theorem.** *Let X be ω-linear with $T^0 = T^1$. If (M, X) is recurrent at each point, then (M, X) partitions into minimal sets.*

Proof. Suppose there is an orbit closure, $\overline{\mathrm{Orb}_X(z)}$, that is not minimal; then there are points x and $y \in \overline{\mathrm{Orb}_X(z)}$ such that $\delta(\mathrm{Orb}_X(y), x) = \alpha > 0$. Let B be a constant such that $\|\hat{X}_t|_{T^1(p)}\| \le B$ for all $t \in R$ and $p \in M$. Pick $\bar{z} = z\bar{T}$ such that $\bar{z} = Y_1^+ \circ Y_1^- \circ Y_1^0(y)$ with $\|Y^0(p)\| \le \alpha/2B$ for all p; then

$$\delta(X_t(y), X_t(Y^0(y))) \le \int_0^1 \|\hat{X}_t(Y^0(Y_s^0(y)))\|\, ds$$
$$\le \int_0^1 B\|Y^0(Y_s^0(y))\|\, ds = B\alpha/2B = \alpha/2$$

for each t in R. Thus $\delta(\mathrm{Orb}_X(Y_1^0(Y)), x) \ge \alpha/2$. By 3.12, $\delta(\mathrm{Orb}_X(Y_1^- \circ Y_1^0(y)), x) \ge \alpha/2$ and $\delta(\mathrm{Orb}_X(Y_1^+ \circ Y_1^- \circ Y_1^0(y)), x) \ge \alpha/2$. Since $\bar{z} = Y_1^+ \circ Y_1^- \circ Y_1^0(y) = z\bar{t}$, $\mathrm{Orb}_X(\bar{z}) = \mathrm{Orb}_X(z)$ and thus $\delta(\mathrm{Orb}_X(\bar{z}), x) = \delta(\mathrm{Orb}_X(z), x) = 0$. This forms a contradiction and the result follows.

The invariance property of T^+ and T^- vector fields discussed in 3.11 holds equally well for the special zero vector fields.

3.15. **Theorem.** *Let X be ω-linear and let $C = \overline{C^0}$ be X-invariant. If (M, X) is recurrent on a dense set of points and Y is a special zero vector field, then C is Y-invariant.*

Proof. Suppose that Y is special zero and $x \in C$ but $z = Y_s(x) \in C'$. Let $B = \{y \in M | \delta(y, z) < \delta(z, C)\}$ and let $\{t_n\}$ be an extensive set such that $X_{t_n}(z) \in B$ for all n. By 3.3, there exist vector fields Y^a near Y such that

$$\lim_{a \to 0} Y^a_{-s}(z) = Y_{-s}(z) \in C^0 \quad \text{and} \quad \lim_{a \to 0} \delta\{X_{-1/a}(z), X_{-1/a}(Y^a_{-s}(z))\} = 0.$$

Pick $\lambda > 0$ such that, for $0 \le a < \lambda$, $Y^a_{-s}(z) \in C^0$ and such that

$$\delta\{X_{-1/a}(z), X_{-1/a}(Y^a_{-s}(z))\} < \tfrac{1}{2}\{\delta(z, C)\}.$$

Pick t_n such that $0 < -1/t_n < \lambda$, then $\delta\{X_{t_n}(z), X_{t_n}(Y_{-s}^{-1/t_n}(z))\} < \tfrac{1}{2}\{\delta(x, C)\}$. Then $X_{t_n}\{Y_{-s}^{-1/t_n}(z)\} \in C'$. This is contradictory since $Y_{-s}^{-1/t_n}(z) \in C^0$.

3.16. **Theorem.** *Let X be ω-linear with M compact and let $A_1 \cdots A_k$ be a basis for the Lie algebra of a compact Lie group H. If (H, M, X) is a bitransformation group which factors to a minimal transformation group on M/H and if $A_1^* \cdots A_k^*$ are special zero vector fields, then (M, H) is minimal.*

Proof. In [3, p. 46], it is shown that $\{\mathrm{Orb}_X(hx) | h \in H\}$ is a partition of M into minimal sets, that $H_x = \{h \in H | hx \in \overline{\mathrm{Orb}_X(x)}\}$ is a closed Lie subgroup and that $\pi\colon M \to M/R$ is a continuous map onto a Hausdororff space (R is the orbit closure relation). Suppose (M, X) is not minimal, then M/R is not a one point space. We choose disjoint nonempty open sets U and U_1 in M/R. The set $C = \overline{\pi^{-1}(U)}$ is then a proper X-invariant subset $C = \overline{C^0}$ of M. By 3.15, C is A_i^*-invariant for $i = 1, \cdots, n$. Each $h \in H$ can be expressed as a product of finitely many factors of the form $e^{A_i t_i}$. Thus, C is invariant under H and H-invariant. But then $C = M$ which contradicts the assumption. The result follows.

3.17. **Theorem.** *Let X be ω-linear and (M, X) be minimal on a compact manifold, M. If X is a special zero vector field, (M, X) is weakly mixing.*

Proof. In [7, p. 366], it is shown that a minimal flow on a compact space is weakly mixing if and only if $S(M, X) = M \times M$. Suppose there is an $x \in M$ and $t \in \mathbf{R}$ such that $(x, xt) \notin S$. Without loss of generality assume $t > 0$. Let $t_0 = \inf\{t \geq 0 | (x, xt) \notin S(M, X)\}$. Clearly, $t_0 > 0$, by 3.2. Since X is special zero in some neighborhood of xt_0, there is an $\epsilon > 0$ such that $\{x(t_0 + t_1), x(t_0 + t_2)\} \in Q(M, X)$ for $-\epsilon < t_1, t_2 < \epsilon$, by 3.2. Choose t_1 such that $-\epsilon < t_1 < \epsilon$ and $t_0 + t_1 > 0$; then $\{x(t_0 + t_1), x(t_0 + \epsilon/2)\}$ is in $Q(M, X) \leq S(M, X)$, by transitivity of $S(M, X)$. This is a contradiction and thus $(x, xt) \in S(M, X)$, for all $x \in M$ and $t \in \mathbf{R}$. If $y \in M$ and $y = \lim_{n \to \infty} xt_n$, $(x, y) = \lim (x, xt) \in S(M, X)$. Thus $S(M, X) = M \times M$.

If (M, X) is volume invariant, the minimality can be reduced to ergodicity.

3.18. **Lemma.** *Let (M, X) be ergodic on a compact manifold, M. If $S(M, X) = M \times M$ and (M, X) is volume invariant, (M, X) is weakly mixing.*

Proof. It is shown in [7, Theorems 2.5 and 3.3] that if (M, X) is not weakly mixing and there exists an invariant probability measure whose support is M and such that each closed invariant set has measure zero or one, then there is a homomorphism of (M, X) onto a nontrivial equicontinuous transformation group. In [3, Theorem 4.18], it is shown that if $S(M, X) = M \times M$, there are no nontrivial equicontinuous homomorphic images of (M, X). Therefore the existence of a measure as described implies (M, X) is weakly mixing. The volume on M can be normalized so that it is an invariant probability measure. It is clearly supported by M. If C is a closed invariant set with $C^0 \neq \emptyset$ then the volume of C is zero. If $C^0 \neq \emptyset$, since (M, X) is ergodic, $C = M$. The result follows.

3.19. **Theorem.** *Let X be ω-linear and let $(M \times M, X)$ be recurrent on a dense set. If there exists a T^+ or T^- vector field $Y \in \mathfrak{O}(M)$ and a point $y \in M$ such that $\overline{\mathrm{Orb}_Y(y)} = M$, then (M, X) is weakly mixing.*

Proof. Suppose that there is a proper X-invariant subset $C = \overline{C^0}$ of $M \times M$. Pick U and V open sets in M and x and y in M such that $(x, y) \in U \times V \subseteq C^0$ and $\overline{\mathrm{Orb}_Y(x)} = \overline{\mathrm{Orb}_Y(y)} = M$. Pick real numbers r and s such that $\{Y_r(x), Y_s(y)\} \in C'$ and pick a recurrent point (p, q) from the nonempty open set $\{Y_r(U) \times Y_s(V)\} \cap C'$. Let $p_1 = Y_{-r}(p)$ and $q_1 = Y_{-s}(q)$; then

$$\delta\{X_t(p, q), X_t(p_1, q_1)\} = \sqrt{\delta\{X_t(p), X_t(p_1)\}^2 + \delta\{X_t(q), X_t(q_1)\}^2},$$

which tends to zero as t tends to $-\infty$. Let $\{t_n\}$ be an extensive set such that $\lim_{n\to-\infty} X_{t_n}(p, q) = (p, q)$; then $\lim_{n\to-\infty} X_{t_n}(p_1, q_1) = \lim_{n\to-\infty} X_{t_n}(p, q) = (p, q) \in C'$. This is a contradiction and the result follows.

4. Examples. We begin this section with a computational lemma.

4.1. Lemma. *Let U be an open set in M and let $X \in \mathcal{O}(M)$. If there are n horizontal vector fields, $Y_1 \cdots Y_n \in \mathfrak{X}(M)$ with $Y_1(x), \cdots, Y_n(x)$ linearly independent for each x, then $[Y_j, X](x) = \Sigma_{i=1}^n a_{ij}(x) \cdot Y_i(x)$ if and only if $\omega\tilde{X}(Y_1(x) \cdots Y_n(x)) = (a_{ij}(x))$ for all x in M.*

The proof is straightforward and will be omitted.

One can always find a connection with respect to which a given vector field is ω-linear. Let $\{U_a: a \in \mathfrak{A}\}$ be a coordinate cover of M such that $X = \partial/\partial x_1$ on each U_a. Identifying $\pi_L^{-1}(U_a)$ with $U_a \times Gl(n)$ by $(\partial/\partial x_1(y) \cdots \partial/\partial x_n(y)) \cdot g = (y, g)$, one defines ω_a on $T(U_a \times Gl(n))$ by $\omega(Z_x, A_g) = dL_{g^{-1}}(g) \cdot A_g$ for (Z_x, A_g) in $T_x(U_a) \times T_g(Gl(n))$. Thus each $\partial/\partial x_i$ is horizontal and $\omega_a X(x, I) = 0$, since $[\partial/\partial x_i, \partial/\partial x_j] = 0$. Let $\{f_a: a \in \mathfrak{A}\}$ be a partition of unity subordinate to $\{\pi_L^{-1}(U_a): a \in \mathfrak{A}\}$ and let $\omega = \Sigma_{a\in\mathfrak{A}} f_a \omega_a$. Thus, $\omega\tilde{X}(u) = \Sigma_{a\in\mathfrak{A}} f_a(u)\omega_a\tilde{X}(u) = 0$. Since $\omega\tilde{X}$ is constantly zero on all of $L(M)$, X is clearly ω-linear. This connection may not be metric. Indeed, if it is metric, X is equicontinuous.

4.2. Lemma. *Let ω be a flat connection on M and let $\{U_a: a \in \mathfrak{A}\}$ be a cover of neighborhoods such that $\pi_L^{-1}(U_a) = U_a \times Gl(n)$ and $\omega(Z_x, A_g) = dL_{g^{-1}}(g) \cdot A_g = g^{-1} \cdot A$. If $\omega\tilde{X}(x, I) = A$ for each $x \in U_a$ and each $a \in \mathfrak{A}$, then X is ω-linear.*

Proof. If $\tilde{X}(x, g) = (X(x), B_g)$, then $\omega\tilde{X}(x, g) = g^{-1}B$ on any $U_a \times Gl(n)$. Since $\tilde{X}(x, g) = g^{-1}Ag$, $B = A \cdot g$ and $\tilde{X}(x, g) = (X(x), (Ag)_g)$. Let $C_t(x, g) = (X_t x, e^{At}g)$, then C_t forms a one-parameter group of local transformations and the velocity vector coincides with $\tilde{X}$ at each point. Thus, $\tilde{X}_t(x, g) = (X_t(x), e^{At}g)$ for small t and $\tilde{X}(\tilde{X}_t(x, g)) = (X(X_t(x)), (Ae^{At}g)_{e^{At}g})$. Clearly, $\omega\tilde{X}(\tilde{X}_t(x, g)) = (e^{At}g)^{-1}A(e^{At}g) = g^{-1}Ag$. Thus $\tilde{X}$ is constant on orbits of $\tilde{X}$ and of constant canonical form.

4.3. Example. *Let $X \in \mathcal{O}(R^n)$ be the vector field corresponding to the system of affine differential equations $dx/dt = Ax + b$ on $\mathbf{R}^n$. If ω is the usual connection on $\mathbf{R}^n$, then X is ω-linear with the canonical form $\omega\tilde{X}$ given by A.*

Proof. If $L(\mathbf{R}^n)$ is identified with $\mathbf{R}^n \times Gl(n)$ by the identification $(\partial/\partial x_1(y), \cdots, \partial/\partial x_n(y))g \leftrightarrow (y, g)$, then $\omega(Z_x, A_g) = g^{-1} \cdot A$. The vector field X is given by $X(x) = \Sigma_{i=1}^n(\Sigma_{j=1}^n A_{ij}x_j + b_i)\partial/\partial x_i(x)$. We have

$$\left[\frac{\partial}{\partial x_j}, X\right] = \left[\frac{\partial}{\partial x_j}, \sum_{i=1}^{n} \frac{dx_i}{dt}\cdot\frac{\partial}{\partial x_i}\right] = \sum_{i=1}^{n} \frac{dx_i}{dt}\left[\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}\right] + \sum_{i=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{dx_i}{dt}\right)\cdot\frac{\partial}{\partial x_i}$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{dx_i}{dt}\right)\cdot\frac{\partial}{\partial x_i} = \sum_{i=1}^{n} A_{ij}\frac{\partial}{\partial x_i},$$

since dx_i/dt takes x to $\Sigma_{k=1}^n A_{ik}x_k$. Thus $[\partial/\partial x_j, X] = \Sigma_{i=1}^n A_{ij}\partial/\partial x_i$ and $\omega\widetilde{X}(x, I) = A$. By 4.2, X is ω-linear.

The following is an example of an ω-linear vector field on $\mathbf{R}^n$ that corresponds to a nonaffine system of differential equations. Thus, it is clear that ω-linear vector fields are not necessarily infinitesimal affine transformations.

4.4. **Example.** Let $X \in \mathcal{O}(\mathbf{R}^2)$ correspond to the system $dx_1/dt = x_2^3/3 + x_2$ and $dx_2/dt = 0$; then $X(x) = (x_2^3/3 + x_2)(\partial/\partial x_1(x))$. Thus $[\partial/\partial x_1, X] = 0$ and $[\partial/\partial x_2, X](x) = (x_2^2 + 1)(\partial/\partial x_1(x))$, so that

$$\omega\widetilde{X}(x, I) = \begin{pmatrix} 0 & x_2^2+1 \\ 0 & 0 \end{pmatrix}.$$

Since $H\widetilde{X}_t(x, I) = (X_t\binom{x_1}{x_2}, I) = (\binom{f(x,t)}{x_2}, I)$ for some function $f\colon R^2 \times R \to R$,

$$\omega\widetilde{X}(H\widetilde{X}_t(x, I)) = \begin{pmatrix} 0 & x_2^2+1 \\ 0 & 0 \end{pmatrix},$$

independent of $t \in \mathbf{R}$ and the canonical form of $\omega\widetilde{X}$ is constantly that of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus X is ω-linear.

4.5. **Example.** Let G be a Lie group with Lie algebra $\mathfrak{g}$. A basis $A_1, \cdots, A_k$ of $\mathfrak{g}$ provides a means of identifying $L(G)$ with $G \times Gl(k)$ as usual. If ω is the left invariant connection given by $\omega(Z_x, A_g) = g^{-1}A$ where $(Z_x, A_g) \in T(G \times Gl(k))$, and if $X = A_1$, then $[A_j, X] = \Sigma_{i=1}^n C_{j1}^i A_i$ and $\omega\widetilde{X}(x, I) = (C_{j1}^i)$, where $\{C_{jk}^i | i, j, k = 1, \cdots, k\}$ are the structure constants of G with respect to $A_1, \cdots, A_k$.

4.6. **Example.** In [1, p. 53], the three dimensional nilmanifold, M is realized as G/H where

$$G = \left\{\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}_{x,y,z \text{ real}}\right\} \quad \text{and} \quad H = \left\{\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}_{x,y,z \text{ integers}}\right\}$$

Letting

$$K = \left\{\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}_{x_1 \text{ and } x_3 \text{ integers}, x_2 \text{ real}}\right\},$$

G/H is the torus G/K fibred by circles K/H. Let X be the vector field on G/H induced from

$$\exp\begin{pmatrix}0 & \alpha & \gamma\\ 0 & 0 & \beta\\ 0 & 0 & 0\end{pmatrix}t$$

on G. It is shown that $(K/H, G/H, X)$ is a bitransformation group and that X_t projects to the rotation flow $((a, b)/Z, t) \to (a + \alpha t, b + \beta t)/Z$ on the torus.

Let Y_2 and Y_3 be the vector fields on M induced from

$$\exp\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}t \quad \text{and} \quad \exp\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1/\alpha\\ 0 & 0 & 0\end{pmatrix}t$$

respectively, then $L(G/H) = G/H \times Gl(3)$ under the identification $(X(x), Y_2(x), Y_3(x)) \cdot g \leftrightarrow (x, g)$ and $[Y_2, X] = 0$, $[Y_3, X] = Y_2$. If ω is the connection given by $\omega(Z_x, A_g) = g^{-1}A$ for each (Z_x, A_g) in $T(G/H \times Gl(3))$, then

$$\omega\tilde{X}(x, I) = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}.$$

By 4.2, X is ω-linear.

In [1] it is shown that X is distal but not equicontinuous and that X is minimal if the corresponding torus flow is minimal. The theory of ω-linear vector fields provides an alternate proof of each of these facts.

We first note that Y_2 generates the tangent space of the fibre at each point and that Y_2 is a special zero vector field. By 3.16, (M, X) is minimal if the torus flow is minimal. Clearly (M, X) is not equicontinuous since $\omega(\tilde{X})$ is not complex diagonalizable.

4.7. **Example.** A curve γ in a Riemannian manifold M is a geodesic if and only if there is a standard horizontal vector field $B(\xi)$ and a frame u such that $\gamma(t) = \pi_L(B(\xi)_t(u))$ for each t. If $u \cdot \xi \in T(M)$, there is a unique geodesic, γ, with $\gamma(0) = \pi_L(u)$ and $\dot{\gamma}(0) = u \cdot \xi$. We note that $\pi_L(B(\xi)_t(u))$ satisfies the conditions, so that $\gamma(t) = \pi_L(B(\xi)_t(u))$ is the unique geodesic determined by $u \cdot \xi$. The geodesic flow $\{X_t \mid t \text{ real}\}$ is given by $X_t(u \cdot \xi) = \dot{\gamma}(t)$ where γ is the unique geodesic determined by $u \cdot \xi$. Thus

$$\begin{aligned} X_t(u \cdot \xi) &= d\pi_L(B(\xi)_t(u)) \cdot B(\xi)(B(\xi)_t(u)) \\ &= B(\xi)_t(u) \cdot \xi \quad \text{for all } t \in \mathbf{R},\ u \in L(M) \text{ and } \xi \in \mathbf{R}^n. \end{aligned}$$

For $e_1 \in S^{n-1}$, the map $\phi_{e_1}\colon O(M) \to UT(M)$ given by $\phi_{e_1}(u) = u \cdot e_1$ is onto, since any unit vector Y is the first entry of some orthonormal frame u. We note that $\phi_{e_1}(u \cdot e_1) = \{ug \in O(M) | ge_1 = e_1\}$. Thus, if $O(n-1)$ is identified with the isotropy subgroup of e_1 in $O(n)$, we have $O(M)/O(n-1) = UT(M)$. If $g \in O(n-1)$, $dRg(u) \cdot B(e_1)(u) = B(ge_1)(u) = B(e_1)(u)$ for each $u \in O(M)$. Thus $(O(n-1), O(M), B(e_1))$ forms a bitransformation group and the map $\phi_{e_1}: (O(M), Be_1) \to (UT(M), X)$ is a transformation group epimorphism.

If $E_{ij} \in gl(n)$ is the matrix

$$\begin{matrix} & \overset{i}{\downarrow} & \overset{j}{\downarrow} \\ i \to & & -1 \\ j \to & 1 & \end{matrix}$$

and M is a compact manifold of constant curvature k, [8, p. 122],

$$\{B(e_2) + \sqrt{|k|}E_{12}^*, B(e_2) - \sqrt{|k|}E_{12}^*, \cdots, B(e_n) - \sqrt{|k|}E_{1n}^*, E_{23}^*, \cdots, E_{n-1,n}^*, B(e_1)\}$$

is a parallelization of $O(M)$. It is shown in [8, p. 21] that, if M has constant curvature k, $[B(e_i), B(e_j)] = kE_{ij}^*$.

If $k \geq 0$, straightforward calculation yields

$$[B(e_i) \pm \sqrt{k}E_{1i}^*, B(e_1)] = \pm\sqrt{k}(B(e_i) \pm \sqrt{k}E_{1i}^*) \quad \text{and} \quad [E_{ij}^*, B(e_1)] = 0$$

for $2 \leq i < j$. Thus

$$\omega(\widetilde{Be_1}(u, I)) = \begin{pmatrix} 0 & -k & & & & & & \\ k & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & -k & & & \\ & & & k & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix} \leftarrow 2n-2$$

where (u, I) is a pair in the trivialization of $T(O(M))$ generated by the given parallelization. The matrix is complex diagonalizable with zero and pure imaginary eigenvalues. If M is compact, $O(M)$ is compact and $(O(M), B(e_1))$ is equicontinuous. Thus (UTM, X) is equicontinuous.

If $k < 0$, $[B(e_i) \pm \sqrt{-k}E_{1i}^*, B(e_1)] = \pm\sqrt{-k}(B(e_i) + \sqrt{-k}E_{1i}^*)$,

$$\omega(\widetilde{B(e_1)}(u, I)) = \begin{pmatrix} -k & & & & & & & & & \\ & -\sqrt{-k} & & & & & & & & \\ & & -\sqrt{-k} & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & -k & & & & & \\ & & & & & -\sqrt{-k} & & & & \\ & & & & & & 0 & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \\ \\ \leftarrow 2n-2 \\ \\ \\ \\ \end{matrix}$$

Clearly $B(e_i) + \sqrt{-k}E^*_{1i}$ is T^+, $B(e_i) - \sqrt{-k}E^*_{1i}$ is T^- and $d\phi_{e_1}(u) \cdot T^0(u)$ is generated by X, since $T^0(u) = \langle E^*_{23}, \cdots, E^*_{n-1,n}, B(e_1)\rangle$. It is well known that there is a volume element on $UT(M)$ under which (UTM, X) is invariant (see [5]). Thus, (UTM, X) is recurrent on a dense set by the Poincaré recurrence theorem (see [12, p. 10]). If x is a recurrent point of (UTM, X) there is a recurrent point of $(O(M), B(e_1))$ in $\phi^{-1}_{e_1}(x)$ (see [9]). Since ϕ_{e_1} is a principal bundle map, each element of $\phi^{-1}_{e_1}(x)$ is recurrent. Thus $(O(M), B(e_1))$ is densely recurrent and by 3.10, $(UT(M), X)$ is ergodic.

Lemma. *Let σ: $M \to L(M)$ be a global cross section of $L(M)$ and let dV be the volume element defined by $dV(\sigma(x) \cdot g) = \det(g)$. If X is ω-linear on M with only zero eigenvalues, then X is volume invariant.*

Proof. Since any n-tuple from $T_x(M)$ can be expressed as $\sigma(x) \cdot g$ for some matrix g, dV is completely defined. As in the proof of 4.2, $\tilde{X}_t(\sigma(x) \cdot g) = \sigma(X_t(x)) \cdot e^{At}g$ for all $t \in R$, $x \in M$ and $g \in gl(n)$. Since e^{At} is triangular with 1 along the diagonal, $\det(e^{At}) = 1$. Thus $dV(\sigma(x) \cdot g) = \det(g) = \det(e^{At}) \cdot \det(g) = \det(e^{At}g) = dV(\sigma(X_t(x))e^{At}g) = dV(\tilde{X}_t(\sigma(x) \cdot g))$. Therefore dV is invariant by $\tilde{X}$ and the volume is invariant under X.

Consider the parallelization $\{B(e_2) + \sqrt{-k}E^*_{12}, B(e_1), B(e_2) - \sqrt{-k}E^*_{12}, B(e_3) + \sqrt{-k}E^*_{13}, E^*_{23}, B(e_3) - \sqrt{-k}E^*_{13}, \cdots, B(e_n) + \sqrt{-k}E^*_{1n}, E^*_{2n}, B(e_n) - \sqrt{-k}E^*_{1n}, E^*_{34}, \cdots, E^*_{n-1,n}\}$. We find that

$$[B(e_1), B(e_2) + \sqrt{-k}E^*_{12}] = \sqrt{-k}(B(e_2) + \sqrt{-k}E^*_{12}),$$

$$[B(e_2) - \sqrt{-k}E^*_{12}, B(e_2) + \sqrt{-k}E^*_{12}] = 2\sqrt{-k}B(e_1),$$

$$[B(e_i) + \sqrt{-k}E^*_{1i}, B(e_2) + \sqrt{-k}E^*_{12}] = 0 \quad \text{for } i = 3, 4, \cdots, n,$$

$$[B(e_i) - \sqrt{-k}E^*_{1i}, B(e_2) + \sqrt{-k}E^*_{12}] = -2kE^*_{2i},$$

$$[E^*_{2i}, B(e_2) + \sqrt{-k}E^*_{12}] = B(e_i) + \sqrt{-k}E^*_{1i}$$

and

$$[E^*_{ij}, B(e_2) + \sqrt{-k}E^*_{12}] = 0 \quad \text{for } 2 < i < j.$$

Thus

$$\widetilde{\omega(B(e_2) + \sqrt{-k}E^*_{12})}(u, I) = \begin{pmatrix} \begin{pmatrix} 0 & \sqrt{-k} & 0 \\ 0 & 0 & 2\sqrt{-k} \\ 0 & 0 & 0 \end{pmatrix} & & & & & \\ & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -2k \\ 0 & 0 & 0 \end{pmatrix} & & & & \\ & & \ddots & & & \\ & & & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -2k \\ 0 & 0 & 0 \end{pmatrix} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} \leftarrow 3n-3$$

We see that $H = B(e_2) + \sqrt{-k}E^*_{12}$ is volume invariant. Again by Poincaré recurrence $(O(M), H)$ is recurrent on a dense set. We note that $B(e_1)$ is a special zero vector field with respect to H. Let $u \in O(M)$ such that $\phi_{e_1}(\overline{\mathrm{Orb}_{B(e_1)}(u)}) = \overline{\mathrm{Orb}_X(\phi_{e_1}(u))} = UT(M)$. Such a frame u exists since $(UT(M), X)$ is ergodic and therefore has a dense orbit.

Let $\{V_i | i = 1, \cdots, \infty\}$ be a countable base for $O(M)$ and let $\underline{V_i} = \phi_{e_1}^{-1}(\phi_{e_1}(V_i))$. For each i, $\overline{\mathrm{Orb}_H(\underline{V_i})}$ is H-invariant and therefore $B(e_1)$-invariant, by 3.15. Pick $u \in \overline{\mathrm{Orb}_H(\underline{V_i})}$ such that $\phi_{e_1}(\overline{\mathrm{Orb}_{B(e_1)}(u)}) = UTM$, then clearly $\overline{\mathrm{Orb}_H(\underline{V_i})}$ meets every fibre for each i. That is $\mathrm{Orb}_H(\underline{V_i}) \cap V_j \neq \emptyset$ for each i

and j. By the Baire theorem, $\bigcap_{i=1}^{\infty} \mathrm{Orb}_H(\underline{V_i}) \neq \emptyset$. Let $v \in \bigcap_{i=1}^{\infty} \mathrm{Orb}_H(\underline{V_i})$, let W be an open set of UTM and let $\underline{V_{i_0}} \subseteq \phi_{e_1}^{-1}(W)$. There exists $t \in R$ and $z \in \underline{V_{i_0}}$ such that $v = H_t(z)$ or $z = H_{-t}(v)$. Thus $\mathrm{Orb}_H(v) \cap \phi_{e_1}^{-1}(W) \neq \emptyset$ and $\phi_{e_1}(\mathrm{Orb}_H(v)) \cap W \neq \emptyset$. Thus $\phi_{e_1}\overline{(\mathrm{Orb}_H(v))} = UTM$.

Since H is T^+ with respect to $B(e_1)$, $\mathrm{Orb}_H(v) \times \mathrm{Orb}_H(v) \subseteq P(O(M), B(e_1))$ and $\phi_{e_1}(\mathrm{Orb}_H(v) \times \mathrm{Orb}_H(v)) \subseteq P(UT(M), X) \subseteq S(UTM, X)$. Since $\phi_{e_1}(\mathrm{Orb}_H(v) \times \mathrm{Orb}_H(v))$ is a dense set of UTM, $S(UT(M), X) = UT(M) \times UT(M)$ and, by 3.18, (UTM, X) is weakly mixing.

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